A Completeness Theorem for a 3-Valued
Semantics for a First-order Language

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This document presents a Gentzen-style deductive calculus and proves that it is complete with respect to a 3-valued semantics for a language with quantifiers. The semantics resembles the strong Kleene semantics with respect to conjunction, disjunction and negation. The completeness proof for the sentential fragment fills in the details of a proof sketched in Arnon Avron (2003) “Classical Gentzen-type Methods in Propositional Many-valued Logics” in Beyond Two: Theory and Application of Multiple-Valued Logics, M. Fitting and E. Orlowska, eds., pp. 117-155. Physica Verlag. The extension to quantifiers is original but uses standard techniques.

Sentential Logic

Let $SL$ be a sentential language with connectives $\neg$, $\rightarrow$, $\land$, $\lor$ and standard syntax. Let $GS3$ be a Gentzen-style deductive calculus with the following rules:

- **(Basis)** $A \Rightarrow A$ i.e. $\emptyset \vdash A$

- **(Weakening)** $\Gamma \Rightarrow \Delta \Rightarrow \Delta, \Delta'$

- **(Cut)** $\frac{\Gamma_1 \Rightarrow \Delta_1, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$

- **($\bot \Rightarrow$)** $\neg A, A \Rightarrow$ i.e. $\emptyset \vdash \neg A, A \Rightarrow \emptyset$

- **($\neg \neg \Rightarrow$)** $A, \Gamma \Rightarrow \Delta \Rightarrow \neg \neg A, \Gamma \Rightarrow \Delta$
(⇒ ¬¬) \[ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg \neg A} \]

(⇒⇒) \[ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \Rightarrow B, \Gamma \Rightarrow \Delta} \]

(⇒⇒) \[ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \Rightarrow B} \]

(¬⇒⇒) \[ \frac{A, \neg B, \Gamma \Rightarrow \Delta}{\neg (A \Rightarrow B), \Gamma \Rightarrow \Delta} \]

(⇒ ¬⇒) \[ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, (A \Rightarrow B)} \]

(∧ ⇒) \[ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \land B \Rightarrow \Delta} \]

(⇒ ∧) \[ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \land B} \]

(¬∧ ⇒) \[ \frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg (A \land B) \Rightarrow \Delta} \]

(⇒ ¬∧) \[ \frac{\Gamma \Rightarrow \Delta, \neg A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg (A \land B)} \]

(∨ ⇒) \[ \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \lor B \Rightarrow \Delta} \]

(⇒ ∨) \[ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \lor B} \]

(¬∨ ⇒) \[ \frac{\Gamma \Rightarrow A, \neg B \Rightarrow \Delta}{\Gamma, \neg (A \lor B) \Rightarrow \Delta} \]

(⇒ ¬∨) \[ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg (A \lor B)} \]

NB: Arbitrary rearrangements of elements before “⇒” and arbitrary rearrangements of elements after “⇒” are allowed.

**Definition.** A *valuation* \( v \) is an assignment to atomic sentences of \( SL \) of members of \( \{Y, I, N\} \) ("yes", "indeterminate" and "no").

**Definition.** \( V \) extends a valuation \( v \) to every sentence of \( SL \) iff for all sentences \( P \) of \( SL \):

1. if \( P \) is atomic: \( V(P) = v(P) \)
2. If $P = \neg Q$, then:
   (a) $V(P) = Y$ if $V(Q) = N$,
   (b) $V(P) = N$ if $V(Q) = Y$,
   (c) $V(P) = I$ otherwise;

3. If $P = (Q \rightarrow R)$, then:
   (a) $V(P) = Y$ if $V(Q) \in \{I, N\}$ or $V(R) = Y$,
   (b) $V(P) = N$ if $V(Q) = Y$ and $V(R) = N$,
   (c) $V(P) = I$ if $V(Q) = Y$ and $V(R) = I$;

4. If $P = (Q \wedge R)$, then:
   (a) $V(P) = Y$ if $V(Q) = V(R) = Y$,
   (b) $V(P) = N$ if $V(Q) = N$ or $V(R) = N$,
   (c) $V(P) = I$ otherwise;

5. If $P = (Q \lor R)$, then:
   (a) $V(P) = Y$ if $V(Q) = Y$ or $V(R) = Y$,
   (b) $V(P) = N$ if $V(Q) = V(R) = N$,
   (c) $V(P) = I$ otherwise;

In tables:

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NB: This set of connectives is not functionally complete. That is, not all truth functions on $\{Y, I, N\}$ can be defined by means of them (Avron 2003, p. 219).

**Definition.** A model for a sequence $\Gamma \Rightarrow \Delta$ is a valuation $v$ s.t. if $V$ extends $v$, then for some $P \in \Gamma, V(P) \in \{I, N\}$ or for some $P \in \Delta, V(P) = Y$.

**Definition.** $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n$ is valid if for every valuation $v$, if $v$ is a model of each of $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n$, then it is a model of $\Gamma \Rightarrow \Delta$. 

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3
Definition. Γ ⊢ Δ if \( \emptyset \Gamma \Rightarrow \Delta \) is valid.

Definition. Where \( S = \{ \Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n \} \), an \( S \)-cut is an application of (Cut) in which \( A \in \left( \bigcup_{i=1}^{n} \Gamma_i \right) \cup \left( \bigcup_{i=1}^{n} \Delta_i \right) \).

Definition. An \( S \)-proof of \( \Delta \) from a set of sequences \( S \) is a proof in which every application of (Cut) is an \( S \)-cut.

Definition. Where \( S = \{ \Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n \} \), \( \Gamma^* \Rightarrow \Delta^* \) is \( S \)-saturated iff:

1. there is no \( S \)-proof of \( \Gamma^* \Rightarrow \Delta^* \);
2. if \( A \in \left( \bigcup_{i=1}^{n} \Gamma_i \right) \cup \left( \bigcup_{i=1}^{n} \Delta_i \right) \) then \( A \in \Gamma^* \cup \Delta^* \);
3. (a) if \( \neg
\neg A \in \Gamma^* \), then \( A \in \Gamma^* \),
   (b) if \( \neg
\neg A \in \Delta^* \), then \( A \in \Delta^* \);
4. (a) if \( A \rightarrow B \in \Gamma^* \), then \( A \in \Delta^* \) or \( B \in \Gamma^* \),
   (b) if \( A \rightarrow B \in \Delta^* \), then \( A \in \Gamma^* \) and \( B \in \Delta^* \),
   (c) if \( \neg(A \rightarrow B) \in \Gamma^* \), then \( A \in \Gamma^* \) and \( \neg B \in \Gamma^* \),
   (d) if \( \neg(A \rightarrow B) \in \Delta^* \), then \( A \in \Delta^* \) or \( \neg B \in \Delta^* \);
5. (a) if \( A \land B \in \Gamma^* \), then \( A \in \Gamma^* \) and \( B \in \Gamma^* \),
   (b) if \( A \land B \in \Delta^* \), then \( A \in \Delta^* \) or \( B \in \Delta^* \),
   (c) if \( \neg(A \land B) \in \Gamma^* \), then \( \neg A \in \Gamma^* \) or \( \neg B \in \Gamma^* \),
   (d) if \( \neg(A \land B) \in \Delta^* \), then \( \neg A \in \Delta^* \) and \( \neg B \in \Delta^* \);
6. (a) if \( A \lor B \in \Gamma^* \), then \( A \in \Gamma^* \) or \( B \in \Gamma^* \),
   (b) if \( A \lor B \in \Delta^* \), then \( A \in \Delta^* \) and \( B \in \Delta^* \),
   (c) if \( \neg(A \lor B) \in \Gamma^* \), then \( \neg A \in \Gamma^* \) and \( \neg B \in \Gamma^* \),
   (d) if \( \neg(A \lor B) \in \Delta^* \), then \( \neg A \in \Delta^* \) or \( \neg B \in \Delta^* \).
Similarly, there is an $S$-proof of $\Gamma \Rightarrow \Delta$. If $\Gamma$, $\Delta$ be sets consisting of formulas that are either subformulas or negations of subformulas in $(\Gamma \cup \Delta) \cup \left( \bigcup_{i=1}^{n} \Gamma_i \right) \cup \left( \bigcup_{i=1}^{n} \Delta_i \right)$.

**Proof.**

1. **Case (a):** $\Gamma \not\subseteq \Delta$. Suppose $\Gamma \not\subseteq \Delta$. Then by (Cut), $\Gamma \not\Rightarrow \Delta$ has an $S$-proof, contrary to assumption.

2. **Case (b):** $\Delta \not\subseteq \Delta$. Since proofs are finite, there is an $i$ such that $\Gamma_i \Rightarrow \Delta_i$ (in our construction) has an $S$-proof, contrary to (i).

**NB:** If $\Gamma^* \Rightarrow \Delta^*$ is $S$-saturated, then membership in $\Gamma^*$ behaves like $\neg$ and membership in $\Delta^*$ behaves like $\not\in$ or $\not\subseteq$.

Let $s_1, s_2, \ldots, s_m, \ldots$ be a list of all formulas that are either subformulas or negations of subformulas in $(\Gamma \cup \Delta) \cup \left( \bigcup_{i=1}^{n} \Gamma_i \right) \cup \left( \bigcup_{i=1}^{n} \Delta_i \right)$.

Construct $\Gamma^* \Rightarrow \Delta^*$ thus:

Let $\Gamma^0 = \Gamma, \Delta^0 = \Delta$.

For all $i \geq 0$, let $\Gamma^{i+1} = \Gamma^i \cup \{s_{i+1}\}, \Delta^{i+1} = \Delta^i$, if there is no $S$-proof of $\Gamma^i, s_{i+1} \Rightarrow \Delta^i$.

For all $i \geq 0$, let $\Gamma^{i+1} = \Gamma, \Delta^{i+1} = \Delta^i \cup \{s_{i+1}\}$, if there is an $S$-proof of $\Gamma^i, s_{i+1} \Rightarrow \Delta^i$.

Let $\Gamma^* = \bigcup_{i=1}^{\infty} \Gamma^i$ and $\Delta^* = \bigcup_{i=1}^{\infty} \Delta^i$.

**Observation 1:** $A \in \Gamma^* \cup \Delta^*$ iff $A$ is a subformula or a negation of a subformula of a formula in $(\Gamma \cup \Delta) \cup \left( \bigcup_{i=1}^{n} \Gamma_i \right) \cup \left( \bigcup_{i=1}^{n} \Delta_i \right)$.

**Lemma 1.** Suppose there is no $S$-proof of $\Gamma \Rightarrow \Delta$. Then:

1. (i) For each $i \geq 0$, $\Gamma^{i+1} \Rightarrow \Delta^{i+1}$ has no $S$-proof.
2. (ii) There is no $S$-proof of $\Gamma^* \Rightarrow \Delta^*$.
3. (iii) Maximality: Let $\Gamma_0$ and $\Delta_0$ be sets consisting of formulas that are either subformulas or negations of subformulas of formulas in $(\Gamma \cup \Delta) \cup \left( \bigcup_{i=1}^{n} \Gamma_i \right) \cup \left( \bigcup_{i=1}^{n} \Delta_i \right)$. If $\Gamma_0 \not\subseteq \Gamma^*$ or $\Delta_0 \not\subseteq \Delta^*$, then $\Gamma_0, \Gamma^* \Rightarrow \Delta^*, \Delta_0$ has an $S$-proof.

**Proof.**

(i) By induction:

**Basis:** By assumption, $\Gamma^0 \Rightarrow \Delta^0$ has no $S$-proof.

**Induction hypothesis:** Suppose $\Gamma^i \Rightarrow \Delta^i$ has no $S$-proof.

**Induction step:** By the construction, either $\Gamma^{i+1} = \Gamma^i \cup \{s_{i+1}\}$ or $\Delta^{i+1} = \Delta^i \cup \{s_{i+1}\}$. If $\Gamma^{i+1} = \Gamma^i \cup \{s_{i+1}\}$, then, by the construction, $\Gamma^{i+1} \Rightarrow \Delta^{i+1}$ has no $S$-proof. If $\Delta^{i+1} = \Delta^i \cup \{s_{i+1}\}$, then there is an $S$-proof of $\Gamma^i, s_{i+1} \Rightarrow \Delta^i$.

Suppose $\Gamma^{i+1} \Rightarrow \Delta^{i+1}, \Delta^i \cup \{s_{i+1}\}$, i.e., $\Gamma^i \Rightarrow \Delta^i, s_{i+1}, \Delta_0$ has an $S$-proof. Then by (Cut), $\Gamma^i \Rightarrow \Delta^i$ has an $S$-proof, contrary to assumption.

(ii) Suppose there is an $S$-proof of $\Gamma^* \Rightarrow \Delta^*$. Since proofs are finite, there is an $i$ such that $\Gamma_i \Rightarrow \Delta_i$ (in our construction) has an $S$-proof, contrary to (i).

(iii) Let $\Gamma_0$ and $\Delta_0$ be as described. **Case (a):** $\Gamma_0 \not\subseteq \Gamma^*$. There is $\Gamma' \subseteq \Gamma_0$ such that $\Gamma' \not\subseteq \Gamma_0$ and $\Gamma' \cap \Gamma^* = \emptyset$. By the construction, $\Gamma'_0 \subseteq \Delta^*$. So by (Weakening), there is an $S$-proof of $\Gamma^0, \Gamma^* \Rightarrow \Delta^*, \Delta_0$. **Case (b):** $\Delta_0 \not\subseteq \Delta^*$. Similarly.

\[\square\]
Lemma 2. If \( \Gamma \Rightarrow \Delta \) has no S-proof, then \( \Gamma^* \Rightarrow \Delta^* \) (as in the construction) is S-saturated.

Proof. Suppose \( \Gamma \Rightarrow \Delta \) has no S-proof.

Condition (1) in the definition of S-saturated: By Lemma 1(ii).

Condition (2) in the definition of S-saturated: Suppose, for a reductio, that \( A \in \left( \bigcup_{i=1}^n \Gamma_i \right) \cup \left( \bigcup_{i=1}^n \Delta_i \right) \) but \( \neg \neg A \notin \Gamma^* \cup \Delta^* \). By Lemma 1, maximality, \( A, \Gamma^* \Rightarrow \Delta^* \) and \( \Gamma^* \Rightarrow \Delta^*, A \) have S-proofs. But \( \Gamma^* \Rightarrow \Delta^*, A \) is an application of (Cut) to a member of \( \left( \bigcup_{i=1}^n \Gamma_i \right) \cup \left( \bigcup_{i=1}^n \Delta_i \right) \). So \( \Gamma^* \Rightarrow \Delta^* \) has an S-proof, contrary to Lemma 1(ii).

Condition (3): (a) Suppose \( \neg \neg A \notin \Gamma^* \). \( A, \Gamma^* \Rightarrow \Delta^* \) is an application of \( (\neg \neg \Rightarrow) \). But \( \neg \neg A, \Gamma^* = \Gamma^* \). So \( \Gamma^* \Rightarrow \Delta^* \) has an S-proof if \( A, \Gamma^* \Rightarrow \Delta^* \) has one. So \( A, \Gamma^* \Rightarrow \Delta^* \) has no S-proof. So \( A \notin \Delta^* \). So by the construction (and Observation 1), \( A \in \Gamma^* \). (b) Suppose \( \neg \neg A \notin \Delta^* \). By \( (\Rightarrow \neg \neg) \), \( \Gamma^* \Rightarrow \Delta^* \) has an S-proof if \( \Gamma^* \Rightarrow \Delta^*, A \) has one. So \( \Gamma^* \Rightarrow \Delta^*, A \) has none. So \( A \notin \Gamma^* \). So \( A \in \Delta^* \).

Condition (4): (a) Suppose \( A \rightarrow B \in \Gamma^* \). \( \Gamma^* \Rightarrow \Delta^*, A \rightarrow B, \Gamma^* \Rightarrow \Delta^* \) is an application of \( (\rightarrow \Rightarrow) \). But \( A \rightarrow B, \Gamma^* = \Gamma^* \). Since \( \Gamma^* \Rightarrow \Delta^* \) lacks an S-proof, either (i) \( \Gamma^* \Rightarrow \Delta^*, A \) has no S-proof, or (ii) \( B, \Gamma^* \Rightarrow \Delta^* \) has no S-proof. Suppose (ii). \( A \notin \Delta^* \). So by the construction, \( A \in \Delta^* \). Suppose (ii).

\( B \notin \Delta^* \). So \( B \in \Gamma^* \). (b) Suppose \( A \rightarrow B \in \Delta^* \). \( A, \Gamma^* \Rightarrow \Delta^*, B \) is an application of \( (\Rightarrow \Rightarrow) \). But \( \Delta^*, A \rightarrow B = \Delta^* \). Since \( \Gamma^* \Rightarrow \Delta^* \) has no S-proof \( \Gamma^*, A \Rightarrow \Delta^*, B \) has no S-proof. So \( A \notin \Delta^*, B \notin \Gamma^* \), which means \( A \in \Gamma^* \) and \( B \in \Delta^* \). (c) Suppose \( \neg (A \rightarrow B) \in \Gamma^* \). \( \neg (A \rightarrow B), \Gamma^* \Rightarrow \Delta^* \) is an application of \( (\neg \rightarrow \Rightarrow) \). But \( \neg (A \rightarrow B), \Gamma^* = \Gamma^* \). So since \( \Gamma^* \Rightarrow \Delta^* \) has no S-proof, \( A, \neg B, \Gamma^* \Rightarrow \Delta^* \) has no S-proof. So \( A \notin \Delta^*, \neg B \notin \Delta^* \), so \( A \in \Gamma^*, \neg B \in \Gamma^* \). (d) Suppose \( \neg (A \rightarrow B) \in \Delta^* \). \( \Gamma^* \Rightarrow \Delta^*, A \rightarrow B, \Gamma^* \Rightarrow \Delta^*, \neg B \) is an application of \( (\Rightarrow \neg \Rightarrow) \). But \( \Delta^*, \neg (A \rightarrow B) = \Delta^* \). Since \( \Gamma^* \Rightarrow \Delta^* \) has no S-proof, either (i) \( \Gamma^* \Rightarrow \Delta^*, A \) has no S-proof, or (ii) \( \Gamma^* \Rightarrow \Delta^*, \neg B \) has no S-proof. So either \( A \notin \Gamma^* \) or \( \neg B \notin \Gamma^* \). So either \( A \in \Delta^* \) or \( \neg B \in \Delta^* \).

Conditions (5) and (6): Similarly.

Lemma 3. If \( \Gamma^* \Rightarrow \Delta^* \), constructed from \( \Gamma \Rightarrow \Delta \), is S-saturated, then there is a valuation that is a model of every sequence in \( S \), but not a model of \( \Gamma \Rightarrow \Delta \).
Proof. Suppose $\Gamma^* \Rightarrow \Delta^*$ is $S$-saturated. Define valuation $v$ as follows: For all atomic formulas $P$ of $SL$, $v(P) = \begin{cases} Y & \text{if } P \in \Gamma^* \\ I & \text{if } P \notin \Gamma^* \text{ and } \neg P \notin \Gamma^* \\ N & \text{if } \neg P \in \Gamma^* \end{cases}$.

First step: $v$ is well-defined: If $P \in \Gamma$ and $\neg P \in \Gamma^*$, then by (Weakening) $\Gamma^* \Rightarrow \Delta^*$ will be an $S$-proof. Since $\Gamma^* \Rightarrow \Delta^*$ (by $S$-satisfaction) does not have an $S$-proof, either $P \notin \Gamma^*$ or $\neg P \notin \Gamma^*$. So the definition of $v$ does not yield both $v(P) = Y$ and $v(P) = N$.

Second step: Suppose $V$ extends $v$. Prove for all formulas $P$ of $SL$, if $P \in \Gamma^*$ then $V(P) = Y$ and if $P \in \Delta^*$ then $V(P) \in \{I, N\}$.

By induction:

Basis: The thesis holds for all literals (atomic sentences and negations of atomic sentences): First, consider atomic $P$. (i) Suppose $P \in \Gamma^*$. $v(P) = V(P) = Y$. (ii) Suppose $P \in \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ has no $S$-proof, $P \notin \Gamma^*$. If $\neg P \notin \Gamma^*$, then $v(P) = V(P) = I \in \{I, N\}$. If $\neg P \in \Gamma^*$, then $v(P) = V(P) = N \in \{I, N\}$. Next, consider $\neg P$, where $P$ is atomic. (i) Suppose $\neg P \in \Gamma^*$. $v(P) = V(P) = N$. $V(\neg P) = Y$. (ii) Suppose $\neg P \in \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ has no $S$-proof, $\neg P \notin \Gamma^*$. If $P \notin \Gamma^*$, then $v(P) = V(P) = I$ and $V(\neg P) = I \in \{I, N\}$. If $P \in \Gamma^*$, then $v(P) = V(P) = Y$ and $V(\neg P) = N \in \{I, N\}$.

Induction hypothesis: The thesis holds for $A, B, \neg A$ and $\neg B$.

Induction step: Show that it holds for $\neg \neg A, (A \rightarrow B), \neg(A \rightarrow B), (A \land B), \neg(A \land B), (A \lor B)$ and $\neg(A \lor B)$.

($\neg \neg$): Suppose $\neg \neg A \in \Gamma^*$. By the definition of $S$-saturation, $A \in \Gamma^*$. By IH, $V(A) = Y$. $V(\neg \neg A) = Y$. Suppose $\neg \neg A \in \Delta^*$. By the definition of $S$-saturation, $A \in \Delta^*$. By IH, $V(A) \in \{I, N\}$. $V(\neg \neg A) \in \{I, N\}$.

($\rightarrow$): Suppose $(A \rightarrow B) \in \Gamma^*$. By the definition of $S$-saturation, $A \in \Delta^*$ or $B \in \Gamma^*$. By IH, $V(A) \in \{I, N\}$, or $V(B) = Y$. $V(A \rightarrow B) = Y$. Suppose $(A \rightarrow B) \in \Delta^*$. By the definition of $S$-saturation, $A \in \Gamma^*$ and $B \in \Delta^*$. By IH, $V(A) = Y, V(B) \in \{I, N\}$. $V(A \rightarrow B) \in \{I, N\}$.

($\neg \rightarrow$) Suppose $\neg(A \rightarrow B) \in \Gamma^*$. By the definition of $S$-saturation, $A \in \Gamma^*$ and $\neg B \in \Gamma^*$. By IH, $V(A) = Y$ and $V(\neg B) = Y$. $V(\neg(A \rightarrow B)) = Y$. Suppose $\neg(A \rightarrow B) \in \Delta^*$. By the definition of $S$-saturation, $A \in \Delta^*$ or $\neg B \in \Delta^*$. By IH, $V(A) \in \{I, N\}$ or $V(\neg B) \in \{I, N\}$, $V(A) \in \{I, N\}$ or $V(B) \in \{Y, I\}$. $V(A \rightarrow B) \in \{Y, I\}$. $V(\neg(A \rightarrow B)) \in \{I, N\}$.

Cases $(\land), (\neg \land), (\lor), (\neg \lor)$ similarly.

Consequently, $v$ is not a model of $\Gamma^* \Rightarrow \Delta^*$. So since (by Observation 1)
Third step: Show that \( v \) is a model of every sequence in \( S \). Let \( \Gamma_i \Rightarrow \Delta_i \) be an arbitrary member of \( S \). Show that \( v \) is a model of \( \Gamma_i \Rightarrow \Delta_i \). Suppose, for reductio, that \( \Gamma_i \subseteq \Gamma^* \) and \( \Delta_i \subseteq \Delta^* \). In that case, by (Weakening), \( \Gamma^* \Rightarrow \Delta^* \) has an \( S \)-proof, \( \Gamma^* \Rightarrow \Delta^* \), contrary to Lemma 1(ii). So, either \( \Gamma_i \not\subseteq \Gamma^* \) or \( \Delta_i \not\subseteq \Delta^* \). But by condition (2) in the definition of \( S \)-saturation (or by Observation 1), \( \Gamma_i \cup \Delta_i \subseteq \Gamma^* \cup \Delta^* \). So either (i) there is \( A \in \Gamma_i \) such that \( A \not\in \Delta^* \), or (ii) there is \( A \in \Delta_i \) such that \( A \not\in \Gamma^* \). In case (i) \( V(A) \in \{I,N\} \). So \( v \) is a model of \( \Gamma_i \Rightarrow \Delta_i \). In case (ii) \( V(A) = Y \). So \( v \) is a model of \( \Gamma_i \Rightarrow \Delta_i \).

Completeness Theorem for GS3: If \( \Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n \) is valid, then, where \( S = \{ \Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n \} \), there is an \( S \)-proof of \( \Gamma \Rightarrow \Delta \) in GS3.

Proof. Suppose there is no \( S \)-proof of \( \Gamma \Rightarrow \Delta \). Then, by Lemma 2, \( \Gamma \Rightarrow \Delta \) can be extended to \( S \)-saturated \( \Gamma^* \Rightarrow \Delta^* \). By Lemma 3, \( \Gamma \Rightarrow \Delta \) is not valid.

Corollary 1. If \( \Gamma \models_{3} \Delta \) then there is a proof in GS3 of \( \Gamma \Rightarrow \Delta \).

Extension of these results to \( QL \)

Suppose that \( SL \) is now a language like \( SL \), defined above, except that the atomic formulas are composed, by the usual syntax, from countably many predicates of each adicity and denumerably many variables and denumerably many individual constants. Let \( QL \) be a language like \( SL \) but containing also, for each (individual) variable \( v \), a quantifier \( \forall v \). \( QL \) has the standard syntax, allowing vacuous quantification, and \( \exists v \) abbreviates \( \neg \forall v \neg \). In any sequence, \( \Gamma \Rightarrow \Delta \), the members of \( \Gamma \cup \Delta \) are sentences, containing no free variables. \( Pn/v \) denotes the result of substituting \( n \) for \( v \) wherever \( v \) occurs free in \( P \). \( Pn/v = P \) if and only if \( v \) is not in \( P \). A sentence of \( QL \) is a formula of \( QL \) containing no free variable.

Let \( GQ3 \) be a Gentzen-style deductive calculus containing all of the rules of \( GS3 \) plus the following:

\[
(\forall \Rightarrow) \quad Pn/v, \Gamma \Rightarrow \Delta \quad \forall v P, \Gamma \Rightarrow \Delta
\]

\[
(\Rightarrow \forall) \quad \Gamma \Rightarrow \Delta, Pn/v \quad \Gamma \Rightarrow \Delta, \forall v P, \text{ where } n \not\in P \text{ and not in any member of } \Gamma \cup \Delta, \text{ i.e. } n \text{ is new, or } v \not\in P,
\]
\(\neg v \Rightarrow \) 
\(\neg Pn/v, \Gamma \Rightarrow \Delta\) where \(n\) is not in \(P\) and not in any member of \(\Gamma \cup \Delta\), 
i.e. \(n\) is new, or \(v\) is not in \(P\),

\(\Rightarrow \neg v\) 
\(\Gamma \Rightarrow \Delta, \neg Pn/v\) 
\(\Gamma \Rightarrow \Delta, \neg \forall vP\)

Define a 3-valued structure \(\mathfrak{M}\) as a triple \((U, \Sigma^+, \Sigma^-)\) where \(U\), the universe, is a nonempty set of objects, and for each individual constant \(n, \Sigma^+(n) = \Sigma^-(n) = \Sigma(n) \in U\). For each \(m\)-ary predicate \(R\), \(\Sigma^+(R) \subseteq U^m, \Sigma^-(R) \subseteq U^m\), and \(\Sigma^+(R) \cap \Sigma^-(R) = \emptyset\).

Let a structure and variable assignment \(\mathfrak{M}_g\) be a quadruple \((U, \Sigma^+, \Sigma^-, g)\) with \(U, \Sigma^+, \Sigma^-\) as before and \(g\) a partial function over some of the variables of \(QL\) such that for each variable \(v\) in the range of \(g\): \(g(v) \in U\).

\(g[v/o]\) is a variable assignment like \(g\) except that \(v\) is in the range of \(g[v/o]\), whether or not \(v\) was in the range of \(g\), and \(g[v/o]\) assigns \(o\) to \(v\) instead of whatever \(g\) assigns to \(v\), if \(v\) is in the range of \(g\) and does not already assign \(o\) to \(v\). \(g_\emptyset\) is the empty variable assignment with an empty range.

Associate with \(\mathfrak{M}_g\) the function \(h\) such that for each singular term \(t\) of \(QL\) that is either an individual constant of \(QL\) or a variable of \(QL\) in the range of \(g\), 
\(h(t) = \begin{cases} 
\Sigma(t) \text{ if } t \text{ is an individual constant,} \\
g(t) \text{ if } t \text{ is a variable.}
\end{cases}\)

A structure \(\mathfrak{M} = \mathfrak{M}_{g_\emptyset}\).

Associate with each structure and variable assignment \(\mathfrak{M}_g\) a function of the same name from formulas of \(QL\) into \(\{Y, I, N\}\), as follows:
\(\mathfrak{M}_g(Rt_1t_2\ldots t_m) = Y\) iff \(\langle h(t_1), h(t_2), \ldots, h(t_m) \rangle \in \Sigma^+(R)\),
\(\mathfrak{M}_g(Rt_1t_2\ldots t_m) = N\) iff \(\langle h(t_1), h(t_2), \ldots, h(t_m) \rangle \in \Sigma^-(R)\),
\(\mathfrak{M}_g(Rt_1t_2\ldots t_m) = I\) otherwise,
\(\mathfrak{M}_g(\neg P) = Y\) iff \(\mathfrak{M}_g(P) = N\),
\(\mathfrak{M}_g(\neg P) = N\) iff \(\mathfrak{M}_g(P) = Y\),
\(\mathfrak{M}_g(\neg P) = I\) otherwise,
\(\mathfrak{M}_g((P \rightarrow Q)) = Y\) if \(\mathfrak{M}_g(P) \in \{I, N\}\) or \(\mathfrak{M}_g(Q) = Y\),
\(\mathfrak{M}_g(P \rightarrow Q) = N\) if \(\mathfrak{M}_g(P) = Y\) and \(\mathfrak{M}_g(Q) = N\),
\(\mathfrak{M}_g((P \rightarrow Q)) = I\) otherwise,
\(\mathfrak{M}_g((P \wedge Q)) = \ldots\) as expected,
\(\mathfrak{M}_g((P \vee Q)) = \ldots\) as expected,
\(\mathfrak{M}_g(\forall vQ) = Y\) iff for all \(o \in U\), \(\mathfrak{M}_{g[v/o]}(Q) = Y\),
\(\mathfrak{M}_g(\forall vQ) = N\) iff for some \(o \in U\), \(\mathfrak{M}_{g[v/o]}(Q) = N\),

9
\( \mathfrak{M}_g(\forall vQ) = I \) otherwise.

Observation 2: If \( U \) is identical to the set of all individual constants of \( QL \) and for all individual constants \( n \) of \( QL \), \( \Sigma(n) = n \), then \( \mathfrak{M}(\forall vQ) = Y \) iff for all \( n \) of \( QL \), \( \mathfrak{M}(Qn/v) = Y \), and \( \mathfrak{M}(\forall vQ) = N \) iff for some \( n \) of \( QL \), \( \mathfrak{M}(Qn/v) = N \).

Observation 3: If \( v \) is not in \( Q \), \( \mathfrak{M}_g(\forall vQ) = Y \) iff \( \mathfrak{M}_g(Q) = Y \) and \( \mathfrak{M}_g(\forall vQ) = N \) iff \( \mathfrak{M}_g(Q) = N \).

Definition. A structure \( \mathfrak{M} \) is a model for \( \Gamma \Rightarrow \Delta \) iff either there is \( P \in \Gamma \) such that \( \mathfrak{M}(P) \in \{I,N\} \) or there is \( P \in \Delta \) such that \( \mathfrak{M}(P) = Y \).

Definition. \( \Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n \) is valid\(_{Q3} \) iff every structure that is a model for each of \( \Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n \) is a model for \( \Gamma \Rightarrow \Delta \).

Let \( C \) be a denumerable set of individual constants not in \( QL \). \( QL^+ \) is \( QL \) supplemented by the individual constants in \( C \). By standard techniques, we associate with each formula \( P \) of \( QL^+ \) having exactly one free variable, two members of \( C \), \( c^+_P \) and \( c^-_P \), called the witnesses for \( P \), having the same birth date, such that for no formula \( Q \) of \( QL^+ \) whose witnesses have that same birth date or an earlier birth date does \( Q \) contain \( c^+_P \) or \( c^-_P \). \( c^+_P \) is the positive witness for \( P \) and \( c^-_P \) is the negative witness for \( P \).

Definition. The Henkin set \( \mathcal{H} \) for \( QL^+ \) is the set of sentences \( Q \) of \( QL^+ \) such that for each formula \( P \) of \( QL^+ \) having at most \( v \) free, \( Q \in \mathcal{H} \) if

1. \( n \) is an individual constant of \( QL^+ \) and \( Q = (\forall vP \rightarrow Pn/v) \) or \( Q = (\neg Pn/v \rightarrow \neg \forall vP) \), or
2. \( v \) is not in \( P \), and \( Q = (P \rightarrow \forall vP) \) or \( Q = (\neg \forall vP \rightarrow \neg P) \), or
3. \( c^+_P \) is the positive witness for \( P \) and \( Q = (Pc^+_P/v \rightarrow \forall vP) \), or
4. \( c^-_P \) is the negative witness for \( P \) and \( Q = (\neg \forall vP \rightarrow \neg Pc^-_P/v) \).

Lemma 4. (a) \( (A \rightarrow B), \Gamma \Rightarrow A \) is provable.
(b) \( (A \rightarrow B), \Gamma \Rightarrow A \) is provable.

Proof. (a)
\[ A \Rightarrow A \text{ (Basis)} \]
\[ \Gamma, A \Rightarrow \Delta, B, A \text{ (Weakening)} \]
\[ \Gamma \Rightarrow \Delta, A, (A \rightarrow B) (\Rightarrow \rightarrow) \]
\[ (A \rightarrow B), \Gamma \Rightarrow \Delta \text{ (Weakening)} \]
\[ (A \rightarrow B), \Gamma \Rightarrow \Delta, A \text{ (Cut)} \]

(b) Similarly.

Lemma 5. \[ \Gamma, (\forall vP \rightarrow Pn/v) \Rightarrow \Delta \]
\[ \Gamma \Rightarrow \Delta \text{ is provable.} \]
\[ \Gamma, (\forall vP \rightarrow \forall vP) \Rightarrow \Delta \]
\[ \Gamma \Rightarrow \Delta \text{ where } n \text{ is new, or } v \text{ is not in } P, \text{ is provable.} \]
\[ \Gamma, (\neg \forall vP \rightarrow \neg Pn/v) \Rightarrow \Delta \]
\[ \Gamma \Rightarrow \Delta \text{ where } n \text{ is new, or } v \text{ is not in } P, \text{ is provable.} \]
\[ \Gamma, (\neg Pn/v \rightarrow \neg \forall vP) \Rightarrow \Delta \]
\[ \Gamma \Rightarrow \Delta \text{ is provable.} \]

Proof. By Lemma 4 and \((\forall \Rightarrow), (\Rightarrow \forall), (\neg \forall \Rightarrow) \) and \((\Rightarrow \neg \forall) \) respectively. For example:
\[ \Gamma, (\neg \forall vP \rightarrow \neg Pn/v) \Rightarrow \Delta \]
\[ \Gamma \Rightarrow \Delta, \neg \forall vP \text{ (Lemma 4)} \]
\[ \neg Pn/v, \Gamma \Rightarrow \Delta \text{ (Lemma 4)} \]
\[ \neg \forall vP, \Gamma \Rightarrow \Delta \text{ (}\neg \forall \Rightarrow\text{)} \]
\[ \Gamma \Rightarrow \Delta \text{ (Cut)} \]

The Elimination Theorem Suppose every sentence in \((\Gamma \cup \Delta) \cup \left( \left( \bigcup_{i=1}^{n} \Gamma_i \right) \cup \left( \bigcup_{i=1}^{n} \Delta_i \right) \right) \) is in QL. Suppose also that
\[ \Gamma_1, \mathcal{H} \Rightarrow \Delta_1, \ldots, \Gamma_n, \mathcal{H} \Rightarrow \Delta_n \]
\[ \Gamma, \mathcal{H} \Rightarrow \Delta \text{ is provable in } GQ3 \text{ for } QL^+. \text{ Then} \]
\[ \Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n \]
\[ \Gamma \Rightarrow \Delta \text{ is provable in } GQ3 \text{ for } QL. \]

Proof. Suppose the hypothesis. Since proofs are finite, there is a finite subset \(J \subseteq \mathcal{H} \) such that \( \Gamma_1, J \Rightarrow \Delta_1, \ldots, \Gamma_n, J \Rightarrow \Delta_n \) is provable. Show:
\[ \Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n \]
\[ \Gamma \Rightarrow \Delta \text{ is provable (in } GQ3 \text{ for } QL). \text{ By induction on the size of } J: \]

Basis: \( J = \emptyset \). Trivial.

Induction Hypothesis: Suppose the thesis holds when \( J \) hast \( m \) members (\( m \geq 0 \)). Show that the thesis holds when \( J \) has \( m + 1 \) members.

Case 1: At least one member \( Q \) of \( J \) is of the form \((\forall vP \rightarrow Pn/v) \) or \((\neg Pn/v \rightarrow \neg \forall vP) \) or \((P \rightarrow \forall vP) \) or \((\neg \forall vP \rightarrow \neg P) \). There is a set \( J' \) such that \( J = J' \cup \{Q\}, Q \notin J' \). By the IH it suffices to show that if
Conjecture 1. A valuation val for $QL^+$ is an assignment of the members of \{\text{Y, I, N}\} to sentences of $QL^+$ that are either quantifications or atomic (i.e., not negations, not conditionals, etc.).

Definition. Val extends val to every sentence of $QL^+$ in accordance with tables given above for SL.

The Henkin Construction Theorem: Suppose val is a valuation for $QL^+$ and Val extends val such that for all $Q \in \mathcal{H}, Val(Q) = Y$. Then we can construct a structure $\mathfrak{M}_{Val}$ for $QL^+$ such that for all sentences $P$ of $QL^+$, $Val(P) = Y$ iff $\mathfrak{M}_{Val}(P) = Y$, and $Val(P) = N$ iff $\mathfrak{M}_{Val}(P) = N$ (By implication: $Val(P) = I$ iff $\mathfrak{M}_{Val}(P) = I$).

Proof. Define $\mathfrak{M}_{Val}$ thus:

1. $U$ is identical to the set of individual constants of $QL^+$.
2. For each individual constant $n$ of $QL^+$: $\Sigma(n) = n$.
3. For each $m$-place predicate $R$ of $QL^+$ ($QL$):
   - $\Sigma^+(R) = \{(n_1, n_2, \ldots, n_m) | val(Rn_1n_2\ldots n_m) = Y\}$.
   - $\Sigma^-(R) = \{(n_1, n_2, \ldots, n_m) | val(Rn_1n_2\ldots n_m) = N\}$.

By induction on the length of sentences:

- **Basis**: Suppose $P$ is atomic, i.e. $P = Rn_1n_2\ldots n_m$.
- **Left-to-right**: Suppose $Val(Rn_1n_2\ldots n_m) = Y$. By the construction of $\mathfrak{M}_{Val}$, $(n_1, n_2, \ldots, n_m) \in \Sigma^+(R)$. By the construction of $\mathfrak{M}_{Val}$, $(\Sigma(n_1), \Sigma(n_2), \ldots, \Sigma(n_m))$,
\(\ldots \Sigma(n_m)) \in \Sigma^+(R)\). So by the definition of \(M\) (as a function), \(M_{Val}(R, n_1, n_2, \ldots, n_m) = Y\). Suppose \(Val(R, n_1, n_2, \ldots, n_m) = N\). By the construction, \((n_1, n_2, \ldots, n_m) \in \Sigma^+(R)\). So \(M_{Val}(R, n_1, n_2, \ldots, n_m) = N\).

**Right-to-left:** Suppose \(M_{Val}(R, n_1, n_2, \ldots, n_m) = Y\). By the definition of \(M\) (as a function), \((\Sigma(n_1), \Sigma(n_2), \ldots, \Sigma(n_m)) \in \Sigma^+(R)\). By the construction of \(M_{Val}\), \(\langle n_1, n_2, \ldots, n_m \rangle \in \Sigma^+(R)\). By the construction of \(M_{Val}\), \(Val(R, n_1, n_2, \ldots, n_m) = Y\). Suppose \(M_{Val}(R, n_1, n_2, \ldots, n_m) = N\). Similarly.

**Induction hypotheses:** Suppose the thesis holds for all sentences having complexity \(k\). Show that it holds for all sentences having complexity \(k+1\).

**Induction step:**

\((\neg): \) Exercise.

\((\rightarrow): \) Suppose \(P = (Q \rightarrow R)\).

**Left-to-right:** Suppose \(Val(Q \rightarrow R) = Y\). By the definition of \(Val\), either \(Val(Q) \in \{I, N\} \) or \(Val(Q) = Y\). By IH, either \(M_{Val}(Q) \in \{I, N\} \) or \(M_{Val}(Q) = Y\). By the definition of \(M\), \(M_{Val}(Q \rightarrow R) = Y\). Suppose \(Val(Q \rightarrow R) = N\). By the definition on \(Val\), \(Val(Q) = Y\) and \(Val(R) = N\). By the definition of \(M\), \(M_{Val}(Q \rightarrow R) = N\).

**Right-to-left:** Exercise.

\((\forall)\): **Left-to-right:** Suppose \(Val(\forall v Q) = Y\). By the definition of \(H\), for all individual constants \(n \) in \(QL^+, Val(\forall v Q \rightarrow Qn/v) = Y\). So by the definition of \(Val\), for all individual constants \(n \) in \(QL^+, Val(Qn/v) = Y\). By the IH, for all individual constants \(n \) in \(QL^+, M_{Val}(Qn/v) = Y\). By Observation 2, \(M_{Val}(\forall v Q) = Y\). Suppose \(Val(\forall v Q) = N\). \(Val(\neg \forall v Q) = Y\). Case 1: \(v\) is not in \(Q\). Then by the construction of \(H\), \(Val(\neg \forall v Q \rightarrow \neg Q) = Y\). By the definition of \(Val\), \(Val(\neg Q) = Y\). By IH, \(M_{Val}(\neg Q) = Y\). By the definition of \(M\), \(M_{Val}(\forall v Q) = N\). By Observation 3, \(M_{Val}(\forall v Q) = N\). Case 2: \(v\) is in \(Q\). Then by the construction of \(H\), \(Val(\neg \forall v Q \rightarrow \neg QcQ/v) = Y\). By the definition of \(Val\), \(Val(\neg QcQ/v) = Y\). By the IH, \(M_{Val}(\neg QcQ/v) = Y\). By the definition of \(M\), \(M_{Val}(\forall v Q) = N\). By the definition of \(M\), \(M_{Val}(\forall v Q) = N\).

**Right-to-left:** Suppose \(M_{Val}(\forall v Q) = Y\). Case 1: \(v\) is not in \(Q\). By Observation 3, \(M_{Val}(Q) = Y\). By IH, \(Val(Q) = Y\). By the construction of \(H\), \(Val(Q \rightarrow \forall v Q) = Y\). \(Val(\forall v Q) = Y\). Case 2: \(v\) is in \(Q\). By Observation 2, for all individual constants \(n \) in \(QL^+, M_{Val}(Qn/v) = Y\). In particular, \(M_{Val}(QcQ/v) = Y\). By IH, \(Val(QcQ/v) = Y\). By the construction of \(H\), \(Val(QcQ/v \rightarrow \forall v Q) = Y\). \(Val(\forall v Q) = Y\). Suppose \(M_{Val}(\forall v Q) = N\). By Observation
2, there is an individual constant \( n \) of \( QL^+ \) such that \( \mathfrak{M}_{Val}(Qn/v) = N \).

By IH, \( Val(Qn/v) = N \). \( Val(\neg Qn/v) = Y \). By the construction of \( \mathcal{H} \), \( Val(\neg Qn/v \to \neg \forall v P) = Y \). \( Val(\forall v P) = Y \). \( Val(\forall v P) = N \). \( \square \)

**Completeness Theorem for GQ3:** If \( \Gamma_1 \equiv \Delta_1, \ldots, \Gamma_n \equiv \Delta_n \) is valid\(_{Q3} \), then \( \Gamma \equiv \Delta \) is provable in GQ3.

**Proof.** Suppose that \( \Gamma_1 \equiv \Delta_1, \ldots, \Gamma_n \equiv \Delta_n \) is not provable in GQ3. By the Elimination Theorem, \( \Gamma_1, \mathcal{H} \equiv \Delta_1, \ldots, \Gamma_n, \mathcal{H} \equiv \Delta_n \) is also not provable in GQ3. So it is also not provable in GS3. By the Completeness Theorem for GS3, \( \Gamma_1, \mathcal{H} \equiv \Delta_1, \ldots, \Gamma_n, \mathcal{H} \equiv \Delta_n \) is not valid\(_3 \). So there is a valuation \( val \) such that \( val \) is a model for each of \( \Gamma_1, \mathcal{H} \equiv \Delta_1, \ldots, \Gamma_n, \mathcal{H} \equiv \Delta_n \), but not a model for \( \Gamma, \mathcal{H} \equiv \Delta \). Since \( val \) is not a model for \( \Gamma, \mathcal{H} \equiv \Delta \), for all \( Q \in \mathcal{H}, Val(Q) = Y \).

By the Henkin Construction Theorem, there is a structure \( \mathfrak{M}_{Val} \), such that for all \( P \) of \( QL^+ \), \( Val(P) \) if and only if \( \mathfrak{M}_{Val}(P) = Y \). So \( \mathfrak{M}_{Val} \) is a model for each of \( \Gamma_1, \equiv \Delta_1, \ldots, \Gamma_n, \equiv \Delta_n \), but not for \( \Gamma \equiv \Delta \). So \( \Gamma_1 \equiv \Delta_1, \ldots, \Gamma_n \equiv \Delta_n \) is not valid\(_{Q3} \). \( \square \)